MA 441, Fall 2012

Assignment 13.

This homework is due *Thursday* Dec 13.

There are total 31 points in this assignment. 28 points is considered 100%. If you go over 28 points, you will get over 100% for this homework (up to 115%) and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge*. Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers sections 7.1, 7.2 (partially) in Bartle–Sherbert.

This is an *extra* assignment, meaning that it only goes to the numerator of your course grade.

(1) [1pt] (Part of 7.1.1) If $I = [0, 4]$, calculate the norms of the following partitions:
   (a) $\mathcal{P}_1 = (0, 1, 2, 4)$,
   (b) $\mathcal{P}_2 = (0, 2, 3, 4)$,
   (c) $\mathcal{P}_3 = (0, 1, 1.5, 2, 3.4, 4)$.

(2) [2pt] (Part of 7.1.2) If $f(x) = x^2$ for $x \in [0, 4]$, calculate the following Riemann sums, where $\dot{\mathcal{P}}_i$ has the same partition points as in the previous problem, and the tags are selected as indicated.
   (a) $\mathcal{P}_1$ with the tags at the left endpoints of the subintervals.
   (b) $\mathcal{P}_2$ with the tags at the right endpoints of the subintervals.

(3) [3pt] (7.1.8) If $f \in \mathcal{R}[a, b]$ and $|f(x)| \le M$ for all $x \in [a, b]$, show that
$$\left| \int_a^b f \right| \le M(b - a).$$
(*Hint:* Show the same inequality for each Riemann sum.)

(4) (a) [3pt] (7.1.9) If $f \in \mathcal{R}[a, b]$ and if $(\dot{\mathcal{P}}_n)$ is any sequence of tagged partitions of $[a, b]$ such that $\|\dot{\mathcal{P}}_n\| \to 0$ as $n \to \infty$, prove that $\int_a^b f = \lim_{n\to\infty} S(f; \dot{\mathcal{P}}_n)$.
   (b) [3pt] (7.1.10) Let $g(x) = 0$ if $x \in [0, 1]$ is rational and $g(x) = 1/x$ if $x \in [0, 1]$ is irrational. Prove that $g \notin \mathcal{R}[0, 1]$. However, show that there exists a sequence $(\dot{\mathcal{P}}_n)$ of tagged partitions of $[a, b]$ such that $\|\dot{\mathcal{P}}_n\| \to 0$ as $n \to \infty$ and $\lim_{n\to\infty} S(g; \dot{\mathcal{P}}_n)$ exists.

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(5) (a) [3pt] (7.1.11) Suppose that f is bounded on $[a, b]$ and that there are two sequences of tagged partitions of $[a, b]$ such that $\|\dot{\mathcal{P}}_n\| \to 0$ and $\|\dot{\mathcal{Q}}_n\| \to 0$ as $n \to \infty$, but such that
$$\lim_{n\to\infty} S(f; \dot{\mathcal{P}}_n) \neq \lim_{n\to\infty} S(f; \dot{\mathcal{Q}}_n).$$
Show that f is not in $\mathcal{R}[a, b]$.

(b) [3pt] (7.1.12) Consider the Dirichlet function, defined by $f(x) = 1$ for $x \in [0, 1]$ rational and $f(x) = 0$ for $x \in [0, 1]$ irrational. Show that f is not Riemann integrable on $[0, 1]$. (Hint: You can use problem 5a.)

(6) (a) [3pt] Suppose that $f : [a, b] \to \mathbb{R}$ and that $f(x) = 0$ except for a finite number of points $c_1, \ldots, c_n$ in $[a, b]$. Prove that $f \in \mathcal{R}[a, b]$ and that $\int\limits_a^b f = 0$. (Hint: In a given partition, how many intervals may have a tag with nonzero value of f?)

(b) [2pt] If $g \in \mathcal{R}[a, b]$ and if $f(x) = g(x)$ except for a finite number of points in $[a, b]$, prove that $f \in \mathcal{R}[a, b]$ and that $\int\limits_a^b f = \int\limits_a^b g$. (*Hint:* Apply the above to $f - g$.)

(7) (a) [3pt] (7.2.8) Suppose that f is continuous on $[a, b]$, that $f(x) \geq 0$ on $[a, b]$ and that $\int_a^b f = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

(b) [2pt] (7.2.9) Show that the continuity assumption in (7a) cannot be dropped.

(8) [3pt] (*Mean Value Thereom for Integrals.*) If f is continuous on $[a, b]$, show that there exists $c \in [a, b]$ such that $\int_a^b f = f(c)(b - a)$.